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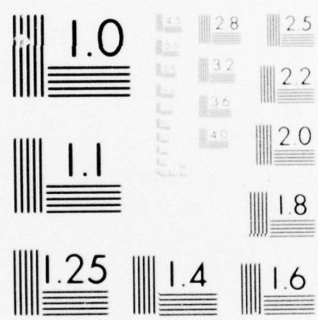

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ON THE OBSERVABILITY OF POLYNOMIAL SYSTEMS

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ABSTRACT

Four natural notions of observability are compared in the case of systems described by polynomial difference equations. The main result states that, for a system having the standard property of (multiple-experiment initial-state) observability, the response to almost any (long-enough) input sequence is sufficient for final-state determination.

1. DEFINITIONS AND PROBLEM STATEMENT

We deal in this paper with discrete-time finite-dimensional nonlinear systems whose next-state and output maps are polynomial functions of previous states and inputs:

$$\Sigma \left\{ \begin{aligned} x(t+1) &= P(x(t),u(t)) \\ y(t) &= h(x(t)), \end{aligned} \right. \qquad t = 0,1,2,\ldots,$$

where $u(t)$, $x(t)$, $y(t)$ belong to algebraic sets U, X, Y over an arbitrary fixed infinite field k, and where $P:X\times U \to X$ and $h:X \to Y$ are polynomial maps. (An <u>algebraic set</u> is a set, in some affine space k^n, defined by polynomial equations $\{ Q_i(x_1, \ldots, x_n) = 0 \}$.) We assume that U is <u>irreducible</u>, i.e. U cannot be expressed as a union of two proper algebraic subsets. (This last restriction is made purely for technical convenience; note that, in particular, any affine space $U = k^m$ is irreducible.)

Such Σ were called <u>polynomial systems</u> in SONTAG and ROUCHALEAU[1975], where various properties were studied using elementary algebraic-geometric tools; all the algebraic-geometric concepts needed for this paper are summarized in that reference.

Denoting the extension of P to input sequences also by $P:X\times U^* \to X$, [for the empty sequence e, $P(x, e) = x$,] we can give the following

(1.1) DEFINITIONS. A polynomial system Σ is:

(a) <u>observable</u> iff for each pair of distinct states x, z in X there exists an input sequence w in U^* such that

$$h(P(x,w)) \neq h(P(z,w)).$$

(One says that w <u>distinguishes</u> between x and z.)

(b) <u>single-experiment observable</u> iff there is some $r \geq 0$ and some (fixed) input sequence $w = (u_1, \ldots, u_r)$ such that for any $x \neq z$ in X,

$$h(P(x,u_1,\ldots,u_t)) \neq h(P(z,u_1,\ldots,u_t))$$

for some $0 \leq t \leq r$.

(c) <u>final-state determinable</u> iff there is some $r \geq 0$ and some (fixed) input sequence $w = (u_1, \ldots, u_r)$ such that

(*) for every pair of states x, z in X, either $h(P(x,u_1,\ldots,u_t)) \neq h(P(z,u_1,\ldots,u_t))$ for some $0 \leq t \leq r$ or $P(x,w) = P(z,w)$.

(d) final-state determinable by generic inputs iff there is some $r \geq 0$ and a proper algebraic set $F \subset U^r$ such that (*) holds for each w not in F.

We wish to prove that the following are precisely the implications that hold:

$$(1.2) \qquad (b) \Rightarrow (a) \Rightarrow (d) \Rightarrow (c).$$

We show below, via counterexamples (1.3),(1.4) and (1.5) respectively, that (a)$\not\Rightarrow$(b), (d)$\not\Rightarrow$(a) and (c)$\not\Rightarrow$(d), while Theorem (2.5) shows that (a)$\Rightarrow$(d).

(1.3) EXAMPLE. Σ_1 has $X := k^2$, $U=Y := k$ and equations

$$\begin{cases} x_1(t+1) = 0, \\ x_2(t+1) = x_1(t)(u(t) - x_1(t)) \\ y(t) = x_2(t). \end{cases}$$

It is not difficult to verify that Σ_1 is observable, in fact even algebraically observable in the sense of SONTAG and ROUCHALEAU[1975]. But no single sequence w serves to distinguish every pair of initial states: let $w = (u,w')$, with u in U; if $u = 0$ then $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\0\end{pmatrix}$ are not distinguished by w, while if $u \neq 0$ then $\begin{pmatrix}u\\0\end{pmatrix}$ is indistinguishable from $\begin{pmatrix}0\\0\end{pmatrix}$.

(1.4) EXAMPLE. Σ_2 has $X = Y := k$, U arbitrary, and equations

$$\begin{cases} x(t+1) = 0 \\ y(t) = 0 . \end{cases}$$

Clearly Σ_2 is not observable, although (d) holds trivially with F = empty, $r = 1$.

(1.5) EXAMPLE. Σ_3 has $X = U = Y := k$ and equations

$$\begin{cases} x(t+1) = x(t)u(t) \\ y(t) = 0 . \end{cases}$$

The sequence (of length one) $w = 0$ solves the final-state determination problem. Comparing any nonzero state x with the zero state shows that any sequence solving the final-state determination problem must have at least one nonzero component, so no proper set F as in (d) can exist.

(1.6) DISCUSSION. (i) Note that (a) is the definition of observability standard in system theory, and that, trivially, (b) implies (a) and (d) implies (c). Property (c) assures the existence of an input sequence w such that, independently of the initial state $x(0)$, the final state $x(r)$ can be determined from the output data $y(0), \ldots, y(r)$. On the other hand, (d) guarantees that such a determination is theoretically possible with essentially any long-enough input sequence, i.e. in real-time system operation. [Of course, this determination is only insured <u>in principle</u>, and effective algorithms inverting the map $(y(0), \ldots, y(r)) \mapsto x(r)$ must still be developed prior to practical applications to nonlinear filtering problems.]

(ii) For linear systems, (a) is equivalent to (b) and (c) is equivalent to (d); see KALMAN, FALB and ARBIB[1969, Chapters 2 and 10].

(iii) The notions corresponding to (a),(b) and (c) have been studied in great detail for finite automata, beginning with MOORE[1956]; see GILL[1962] or CONWAY[1971]. In the automata literature one calls (a) a "diagnosing" and (c) a "homing" problem, and one has precisely the implications (b)$\Rightarrow$(a)$\Rightarrow$(c). [The automata case takes care of polynomial systems over a finite field k; this allows us to restrict attention to infinite k, with the notational advantage of allowing identification of polynomials and polynomial functions.] Finiteness of the state set is crucial in the arguments involved in the automata results, but there appears to be no way of generalizing such arguments through a replacement of finiteness by finite-dimensionality: it can be shown, for instance, that in the context of systems defined by analytic (rather than polynomial) equations, (a) does no longer imply (c), say over k = reals. Thus, one needs to develop completely new arguments here.

(iv) For continuous-time systems, time-reversibility of differential equations implies that (b) is equivalent to (c); this helps perhaps to understand intuitively why (a),(b),(c) may be equivalent in that context, as shown recently by GRASSELLI and ISIDORI[1977] for the case of internally-bilinear continuous-time systems.

2. PROOF OF THE MAIN RESULT

(2.1) LEMMA. For any polynomial system Σ there exists an integer $r \geq 0$ and a proper algebraic set $F \subset U^r$ such that, for every $w = (u_1, \ldots, u_r)$ not in F, and for any x, z in X,

$$h(P(x,u_1,\ldots,u_t)) = h(P(z,u_1,\ldots,u_t)), \quad t = 0,\ldots, r$$

implies

$$P(x,w) \text{ is indistinguishable from } P(z,w).$$

PROOF. Since $Y \subseteq k^p$ for some integer p and since a union of proper algebraic subsets of U^r is again a proper algebraic subset, it is sufficient to prove the lemma with $Y = k$.

Let $s \geq 0$ be such that any pair of distinguishable states is already distinguished by inputs of length $\leq s$ (SONTAG and ROUCHALEAU[1975, Corollary 7.3]).

For any algebraic set Z, let $A(Z)$ denote the algebra of polynomial functions on Z. Let D be the direct limit of the sequence of k-algebras

$$A(U) \to \ldots \to A(U^t) \to A(U^{t+1}) \to \ldots,$$

where

$$A(U) \to A(U^{t+1}) = A(U^t) \otimes A(U): f \mapsto f \otimes 1.$$

Let K be the quotient field of D (which is an integral domain because U is irreducible).

Since $Y = k$, a polynomial map $X \times X \times U^t \to Y$ is an element of $A(X \times X) \otimes A(U^t)$; in particular the functions h_t defined by

$$h_t(x,z,u_1,\ldots,u_t) := h(P(x,u_1,\ldots,u_t)) - h(P(z,u_1,\ldots,u_t))$$

are in $A(X \times X) \otimes K$. The latter is a finitely generated algebra over the field K, hence Noetherian. Thus there is some integer r such that all h_t are in the ideal of $A(X \times X) \otimes K$ generated by $h_o, \ldots, h_r$. In particular, there are therefore equations

$$(2.2) \qquad ch_{r+j} = \sum_{t=0}^{r} a_{jt} h_t, \quad j = 1, \ldots, s,$$

with all a_{jt} in $A(X \times X) \otimes D$ and c a nonzero element of D. Since D is the union of the $A(U^t)$, there is some integer q such that all a_{jt} are in $A(X \times X) \otimes A(U^q)$ and c is in $A(U^q)$. Without loss of generality, we shall assume that $q \geq r + s$.

Define the proper algebraic set

$$F := \left\{ (u_1, \ldots, u_r) \text{ in } U^r \text{ such that } c(u_1, \ldots, u_r, \ldots, u_q) = 0 \text{ for all } (u_{r+1}, \ldots, u_q) \right\}.$$

Claim: F satisfies the requirements of the lemma. Indeed, assume that $\underline{w} = (u_1, \ldots, u_r)$ is not in F. Take x, z in X such that $h(P(x, u_1, \ldots, u_t)) = h(P(z, u_1, \ldots, u_t))$ for all $t = 0, \ldots, r$, i.e.,

$$(2.3) \qquad h_t(x, z, u_1, \ldots, u_t) = 0, \quad t = 0, \ldots, r.$$

Denote $\underline{x} := P(x, \underline{w})$, $\underline{z} := P(z, \underline{w})$. It must be proved that $\underline{x}$, $\underline{z}$ are indistinguishable.

Assume that $\underline{x}$, $\underline{z}$ are distinguished by an input sequence v, which can be taken of length j, $0 \leq j \leq s$, by definition of s. Let

$$F_1 := \left\{ w \text{ in } U^j \text{ such that } h_{r+j}(x, z, \underline{w}, w) = 0 \right\} ;$$

this is an algebraic set, proper because v is not in F_1. Let

$$F_2 := \left\{ w \text{ in } U^j \text{ such that } c(\underline{w}, w, w') = 0 \text{ for all } w' \text{ in } U^{q-r-j} \right\} ;$$

this is also an algebraic set, and it is proper because $\underline{w}$ was taken not in F_2.

It follows that $F_1 \cup F_2$ is also a proper algebraic set. Let then w be in neither F_1 nor F_2. Then $c(\underline{w}, w, w') \neq 0$ for some w', so

$$(2.4) \qquad c(\underline{w}, w, w') h_{r+j}(x, z, \underline{w}, w) \neq 0.$$

But (2.2), (2.3) and (2.4) taken together are contradictory. □

(2.5) THEOREM. <u>Observability implies, for polynomial systems, final-state determinability with generic inputs</u>.

PROOF. Immediate from the lemma. □

(2.6) REMARK. As shown in SONTAG[1976], canonical realizations Σ_f of polynomial response maps are not, in general, polynomial systems. So the Theorem above is not applicable directly ($A(X_f)$ is not Noetherian). However, if f admits a polynomial

realization Σ, then the reachable states of Σ_f form a set which is a quotient of the reachable states of Σ. Then lemma (2.1) can be applied to Σ, implying that the reachable part of Σ_f does satisfy (2.2).

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